J. Fluid Mech. (2003), vol. 490, pp. 333–344.
DOI: 10.1017/S0022112003005329 Printed in the United Kingdom

Closed-form linear stability conditions for magneto-convection

By R. C. KLOOSTERZIEL[1] AND G. F. CARNEVALE[2]

[1]School of Ocean & Earth Science & Technology, University of Hawaii, Honolulu, HI 96822, USA

[2]Scripps Institution of Oceanography, University of California, San Diego, La Jolla, CA 92093, USA

(Received 17 January 2003 and in revised form 9 April 2003)

Chandrasekhar (1961) extensively investigated the linear dynamics of Rayleigh–Bénard convection in an electrically conducting fluid exposed to a uniform vertical magnetic field and enclosed by rigid, stress-free, upper and lower boundaries. He determined the marginal stability boundary and critical horizontal wavenumbers for the onset of convection as a function of the Chandrasekhar number Q or Hartmann number squared. No closed-form formulae appeared to exist and the results were tabulated numerically. We have discovered simple expressions that concisely describe the stability properties of the system. When the Prandtl number Pr is greater than or equal to the magnetic Prandtl number Pm the marginal stability boundary is described by the curve $Q = \pi^{-2}[R - R_c^{1/3} R^{2/3}]$ where R is the Rayleigh number and $R_c = (27/4)\pi^4$ is Rayleigh's famous critical value for the onset of stationary convection in the absence of a magnetic field ($Q = 0$). When $Pm > Pr$ the marginal stability boundary is determined by this curve until intersected by the curve

$$Q = \frac{1}{\pi^2}\left[\frac{Pm^2(1+Pr)}{Pr^2(1+Pm)}R - \left(\frac{(1+Pr)(Pr+Pm)}{Pr^2}\right)^{1/3}\left(\frac{Pm^2(1+Pr)}{Pr^2(1+Pm)}\right)^{2/3} R_c^{1/3} R^{2/3}\right].$$

An expression for the intersection point is derived and also for the critical horizontal wavenumbers for which instability sets in along the marginal stability boundary either as stationary convection or in an oscillatory fashion. A simple formula is derived for the frequency of the oscillations. Also we show that in the limit of vanishing magnetic diffusivity, or infinite electrical conductivity, the system is unstable for sufficiently large R. Instability in this limit always sets in via overstability.

1. Introduction

Thompson (1951) and Chandrasekhar (1952) were the first to consider Rayleigh–Bénard (RB) convection of electrically conducting fluids (i.e. liquid metals) in the presence of an external uniform magnetic field. Here we revisit this classical linear RB-problem with stress-free, rigid upper and lower boundaries. The top and bottom are considered perfect heat conductors, maintained at constant (different) temperatures, with the higher temperature at the bottom. The horizontal domain is unbounded, the Boussinesq approximation is made, incompressibility is assumed and the kinematic viscosity, diffusivity, electrical conductivity and magnetic diffusivity are constant. This work was inspired by our recent success in establishing closed-form linear stability conditions for Rayleigh-Bénard convection subjected to Coriolis forces (see Kloosterziel & Carnevale 2003). Although there are some analogies between the two cases, the stability properties are quite different. However, no comparison will be made here. In § 2.1 we briefly discuss the general properties of the cubic polynomial which determines the eigenvalues p for normal-mode perturbations with an assumed time dependence $\exp(pt)$. In §§ 2.2–2.4 we show that the linear stability results tabulated in chapter IV of Chandrasekhar's (1961) monograph for this 'easy' case of so-called free–free boundaries in the vertical can be summarized with simple closed-form formulae. In § 2.5 we discuss the limit of zero magnetic diffusivity or infinite electrical conductivity. In § 3 we summarize the results and discuss some additional matters of possible interest.

2. Linear stability

In the standard RB-problem density variations are caused by temperature variations. In the unperturbed system, the temperature distribution between the bottom at $z = 0$ and top at $z = d$, with z the vertical coordinate, is linear with a constant adverse temperature gradient $\beta > 0$, so that the temperature at the bottom is higher than at the top. Density is assumed to vary with temperature via a constant coefficient of volume expansion α. The fluid is electrically conducting and subjected to a uniform external vertical magnetic field H which coincides with the direction of the gravitational acceleration g, taken along the z-axis. The equations governing the RB-problem for an incompressible fluid under the Boussinesq approximation are discussed by Chandrasekhar (1961). Linearizing the equations about the basic motionless state, a set of coupled linear equations is derived for the evolution of small-amplitude velocity and temperature perturbations and the accompanying magnetic field perturbations and electric current density. Equations (101), (102) and (105) in chapter IV, § 42, can be used to derive a single equation for the vertical component w of the velocity perturbations. This is a third-order differential equation with respect to time t. The stability of the system is investigated by introducing perturbations $w \propto \exp[pt + \mathrm{i}(k_x x + k_y y)] \sin(n\pi z/d)$ which is the correct form for the boundary conditions at the top and bottom mentioned above (i.e. rigid and stress free). The vertical wavenumber takes the values $n = 1, 2, \ldots$. Substitution of w of the form given above in the third-order equation just mentioned yields a cubic polynomial with the exponential time factor p as variable:

$$\tilde{p}^3 + B\tilde{p}^2 + C\tilde{p} + D = 0, \quad \text{where} \quad \tilde{p} = (d^2/\nu)p \tag{2.1}$$

and

$$\left.\begin{aligned} B &= \frac{Pr + Pm + Pr\,Pm}{Pr\,Pm}(a^2 + n^2\pi^2), \\ C &= \frac{1}{Pr\,Pm}\left[(1 + Pr + Pm)(a^2 + n^2\pi^2)^2 + n^2\pi^2 Pr\,Q - \frac{a^2 Pm R}{a^2 + n^2\pi^2}\right], \\ D &= \frac{1}{Pr\,Pm}[(a^2 + n^2\pi^2)^3 + (a^2 + n^2\pi^2)n^2\pi^2 Q - a^2 R]. \end{aligned}\right\} \tag{2.2}$$

Here

$$R = \frac{g\alpha\beta d^4}{\kappa\nu}, \quad Q = \frac{\sigma\mu^2 H^2 d^2}{\rho_0\nu}, \quad Pr = \frac{\nu}{\kappa} \quad \text{and} \quad Pm = \frac{\nu}{\eta} \tag{2.3}$$

are the Rayleigh number, the Chandrasekhar number (or Hartmann number squared), the Prandtl number and the magnetic Prandtl number, respectively, and

$$a = \left(k_x^2 + k_y^2\right)^{1/2} d$$

is the non-dimensional horizontal wavenumber. Note that p has been non-dimensionalized with the time scale d^2/ν where ν is the kinematic viscosity. ρ_0 is the constant reference density of the fluid which arises in the context of the Boussinesq approximation. κ is the coefficient of thermal diffusivity, σ the electrical conductivity, η the magnetic diffusivity and μ the magnetic permeability. σ, η and μ obey the relation $\sigma = (\mu\eta)^{-1}$.

R, Q, Pr and Pm characterize the system and a and n the perturbations. If for given $\{R, Q, Pr, Pm\}$ for all perturbations the three roots of (2.1) have $\mathrm{Re}\,\tilde{p} < 0$ there is stability. When for certain perturbations there is at least one root with $\mathrm{Re}\,\tilde{p} > 0$, there is instability. With each root of the cubic there is an associated combination of a flow field, temperature and current distribution and a magnetic field, which we refer to as 'modes', although not explicitly considered here. The hyper-surface in the space spanned by $\{R, Q, Pr, Pm\}$ separating stable systems from unstable systems defines the marginally stable states. As explained by Chandrasekhar (1961), when crossing from the stable to the unstable side, instability can set in either as stationary convection in which case one root of (2.1) is $\tilde{p} = 0$, or in an oscillatory fashion when there are two purely imaginary, complex conjugate roots. This is referred to as 'overstability' and the associated modes are called 'overstable modes'. Modes associated with $\tilde{p} = 0$ are called 'convective modes'.

2.1. *The eigenvalues*

It is important to note that B in (2.2) is always positive (assuming non-zero Pr and Pm). As discussed in detail by Kloosterziel & Carnevale (2003) positive B implies that the marginal stability boundary can be determined by mere examination of the coefficients B, C and D without actually solving the cubic: stability/instability is entirely determined by the signs of D and $BC - D$. Since the coefficients B, C and D in (2.1) are real, either the three roots are real or one root is real and the other two are complex conjugates. When $D < 0$ there will be a positive real root, which implies instability. The other two roots always have $\mathrm{Re}\,\tilde{p} < 0$, that is, they are either both negative real or, when complex conjugates, their real part is negative. When $D = 0$, $\tilde{p} = 0$ is a root so that there is at least one convective mode while when $D > 0$ there will be a negative real root and therefore at least one damped mode. For $D \geqslant 0$ the properties of the other two roots are determined by the sign of $BC - D$. The details are given in table 1, where we list the signs of the real part of each root of the cubic for all combinations of D and $BC - D$. In this table 'unstable' simply means that at least one root has $\mathrm{Re}\,\tilde{p} > 0$. 'Stable' means that all three roots have $\mathrm{Re}\,\tilde{p} < 0$, i.e. all modes are damped. This only occurs when both $D > 0$ and $BC - D > 0$. 'Convection' indicates that at least one root is $\tilde{p} = 0$ while none of the other two roots has $\mathrm{Re}\,\tilde{p} > 0$ nor are they purely imaginary. A complex conjugate pair of purely imaginary roots only occurs when $D > 0$ and $BC - D = 0$. That has been indicated with 'overstable'. In the overstable case the non-dimensional frequency ω is determined by $\omega^2 = C = D/B$ (see Kloosterziel & Carnevale 2003).

2.2. *The convection curve*

Equation (2.2) shows that for given a, n and Q, we have $D < 0$ when R is large enough, which according to table 1 implies instability. For small enough R on the other hand $D > 0$, while $D = 0$ when

$$(x + n^2\pi^2)^3 + (x + n^2\pi^2)n^2\pi^2 Q - xR = 0 \quad \text{where} \quad x \equiv a^2 \tag{2.4}$$

	$D<0$	$D=0$	$D>0$
$BC-D<0$	$\tilde{p}>0$ $\mathrm{Re}\,\tilde{p}<0$ $\mathrm{Re}\,\tilde{p}<0$ unstable	$\tilde{p}>0$ $\tilde{p}=0$ $\tilde{p}<0$ unstable	$\tilde{p}<0$ $\mathrm{Re}\,\tilde{p}>0$ $\mathrm{Re}\,\tilde{p}>0$ unstable
$BC-D=0$	$\tilde{p}>0$ $\mathrm{Re}\,\tilde{p}<0$ $\mathrm{Re}\,\tilde{p}<0$ unstable	$\tilde{p}=0$ $\tilde{p}=0$ $\tilde{p}<0$ convection	$\tilde{p}<0$ $\tilde{p}=+\mathrm{i}\omega$ $\tilde{p}=-\mathrm{i}\omega$ overstable
$BC-D>0$	$\tilde{p}>0$ $\mathrm{Re}\,\tilde{p}<0$ $\mathrm{Re}\,\tilde{p}<0$ unstable	$\tilde{p}=0$ $\mathrm{Re}\,\tilde{p}<0$ $\mathrm{Re}\,\tilde{p}<0$ convection	$\tilde{p}<0$ $\mathrm{Re}\,\tilde{p}<0$ $\mathrm{Re}\,\tilde{p}<0$ stable

TABLE 1. Diagram summarizing the properties of the three roots of the cubic (2.1) when $B>0$. Whenever the sign of $\mathrm{Re}\,\tilde{p}$ is given, $\tilde{p}$ can either be real or is one of a complex conjugate pair. In the overstable case the frequency ω is determined by $\omega^2=C$ or $\omega^2=D/B$.

or, alternatively, when

$$R=\frac{x+n^2\pi^2}{x}[(x+n^2\pi^2)^2+n^2\pi^2Q]. \tag{2.5}$$

As a function of x this has a minimum where $\partial_x R=0$, which yields

$$2x^3+3n^2\pi^2x^2=(n^2\pi^2)^3+(n^2\pi^2)^2Q. \tag{2.6}$$

Chandrasekhar (1961) numerically determined the positive real root x_c of this cubic as a function of Q, setting $n=1$ since this yields the smallest critical Rayleigh number. Substituting $x=x_c$ back in (2.5) he obtained the critical Rayleigh number $R_c^{(c)}(Q)$. This is how table XIV with $R_c^{(c)}(Q)$ and $a_c^{(c)}(Q)=x_c^{1/2}$ values in chapter IV of Chandrasekhar (1961) was compiled.

An alternative approach that makes analytical progress possible is the following (see figure 1a). The condition $D=0$ or (2.4) can also be written as

$$f(z)=-n^2\pi^2R \quad \text{where} \quad f(z)=z^3+(n^2\pi^2Q-R)z \quad \text{and} \quad z=x+n^2\pi^2. \tag{2.7}$$

Since $\mathrm{d}_z f=3z^2+(n^2\pi^2Q-R)$, it follows that $\mathrm{d}_z f\geqslant 0$ for all z when $n^2\pi^2Q\geqslant R$, which implies that (2.7) can only be satisfied for one negative z-value. Since $x\geqslant 0$, only $z>0$ are relevant and in this case $D>0$ for all $x\geqslant 0$. However, when $n^2\pi^2Q<R$ there is a local minimum and a maximum, i.e. points where $\mathrm{d}_z f=0$, at

$$z_\pm=\pm\left(\frac{R-n^2\pi^2Q}{3}\right)^{1/2} \quad \text{and} \quad f(z_\pm)=\mp(4/27)^{1/2}(R-n^2\pi^2Q)^{3/2}. \tag{2.8}$$

If $f(z_+)=-n^2\pi^2R$, then the curve $f(z)$ is tangent to the horizontal line $-n^2\pi^2R$ at $z=z_+$ (see figure 1a) and the square of the critical horizontal wavenumber is simply $x_c=z_+-n^2\pi^2$. Solving $f(z_+)=-n^2\pi^2R$, we find that this occurs when

$$Q=Q_c^{(c)}(R,n)=\frac{1}{n^2\pi^2}\left[R-R_c^{1/3}(n)R^{2/3}\right], \tag{2.9}$$

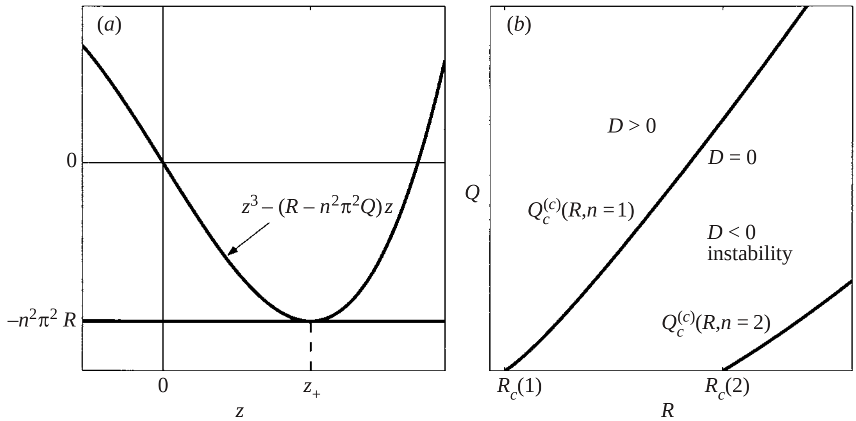

FIGURE 1. (*a*) The critical Chandrasekhar number Q_c and corresponding critical wavenumber $a_c = x_c^{1/2}$ are determined by the condition that the curve $f(z) = z^3 - (R - n^2\pi^2 Q)z$ is tangent to the horizontal line $-n^2\pi^2 R$ at $z = z_+$ with $z = x + n^2\pi^2$ and z_+ the positive z-value for which $\mathrm{d}_z f(z) = 0$ (see text). (*b*) The critical Chandrasekhar number $Q_c^{(c)}$ as a function of R for $n = 1$ and $n = 2$. The leftmost curve $Q_c^{(c)}(R, n = 1)$, given by (2.11), starts on the R-axis at $R = R_c = (27/4)\pi^4$ and for $n > 1$ at $R_c(n) = n^4 R_c$. For $\{R, Q\}$ left of the curve $Q_c^{(c)}(R, n = 1)$ $D > 0$ for all $\{a, n\}$. On the curve, $D = 0$ only when $\{a, n\} = \{a_c^{(c)}(R), 1\}$, with $a_c^{(c)}(R)$ given by (2.12), and $D > 0$ for all other $\{a, n\}$. For points $\{R, Q\}$ to the right of the leftmost curve there are $\{a, n\}$ for which $D < 0$, implying instability, e.g. there will be a positive real root of (2.1).

where $R_c(n) = (27/4)n^4\pi^4$. That is, for fixed R, this is the critical Q-value for which $D = 0$ at $z = z_+ = x_c + n^2\pi^2$, but $D > 0$ for all other $z > 0$. For smaller Q there is a range of positive z-values for which $D < 0$. Substituting Q given by (2.9) back in the expression for z_+ (2.8), we find that

$$x_c = (n^2\pi^2 R/2)^{1/3} - n^2\pi^2. \tag{2.10}$$

In the (R, Q)-plane the curves $Q_c^{(c)}(R, n)$ start on the R-axis (where $Q = 0$) at $R = R_c(n)$. The leftmost curve has $n = 1$, as shown in figure 1(*b*). It follows that $D > 0$ for all $\{a, n\}$ when $\{R, Q\}$ is to the left of the curve

$$Q_c^{(c)}(R) = \frac{1}{\pi^2}\left[R - R_c^{1/3} R^{2/3}\right], \tag{2.11}$$

where $R_c = R_c(n = 1) = (27/4)\pi^4$. This is the inverse of the (implicit) relation $R_c^{(c)}(Q)$ numerically determined by Chandrasekhar (1961). When $\{R, Q\}$ lies exactly on the curve (2.11), $D = 0$ for $\{a, n\} = \{a_c^{(c)}(R), 1\}$ with

$$a_c^{(c)}(R) = \left[(\pi^2 R/2)^{1/3} - \pi^2\right]^{1/2}, \tag{2.12}$$

which follows from (2.10) with $n = 1$ (remember that $x = a^2$), but $D > 0$ for all other $\{a, n\}$. Thus, when $\{R, Q\}$ lies on (2.11) there is one convective mode ($\tilde{p} = 0$) when $\{a, n\} = \{a_c^{(c)}(R), 1\}$ while for all other $\{a, n\}$ at least one mode is damped ($\tilde{p} < 0$; see table 1). For $\{R, Q\}$ to the right of the curve (2.11) there are a and n for which $D < 0$ and there will be unstable modes ($\tilde{p} > 0$; see table 1).

It may be noted that according to (2.11) $Q_c^{(c)} = 0$ when $R = R_c$. Putting $R = R_c$ in (2.12), one finds that $a_c^{(c)}(R_c) = (\pi^2/2)^{1/2}$. $R = R_c$ and $a_c^{(c)} = (\pi^2/2)^{1/2}$ are the famous critical Rayleigh number and horizontal wavenumber found by Rayleigh (1916) in the classical problem without a magnetic field ($Q = 0$) or any other external forces.

2.3. *The overstability curve*

When $\{R, Q\}$ lies to the left of the 'convection curve' (2.11) stability is not guaranteed. According to table 1 we have to investigate the sign of $BC-D$. It follows with (2.2) that $BC-D=0$ when

$$(x+n^2\pi^2)^3+\frac{(x+n^2\pi^2)n^2\pi^2Pr^2Q}{(1+Pr)(Pr+Pm)}-\frac{xPm^2R}{(1+Pm)(Pr+Pm)}=0. \qquad (2.13)$$

Note that (2.13) follows from (2.4) by substituting

$$\frac{Pm^2R}{(1+Pm)(Pr+Pm)} \quad \text{for} \quad R, \quad \text{and} \quad \frac{Pr^2Q}{(1+Pr)(Pr+Pm)} \quad \text{for} \quad Q \qquad (2.14)$$

in (2.4). Thus, the critical wavenumber and Chandrasekhar number for which $BC-D=0$ can be found in an analogous fashion as for $D=0$ by redefining R and Q in (2.7) and (2.8), and then solving for $f(z'_+)=-n^2\pi^2R'$, where $f(z)=z^3+(n^2\pi^2Q'-R')z$ and

$$z'_+=\left(\frac{R'-n^2\pi^2Q'}{3}\right)^{1/2}, \quad R'=\frac{Pm^2R}{(1+Pm)(Pr+Pm)}, \quad Q'=\frac{Pr^2Q}{(1+Pr)(Pr+Pm)}.$$

We then find that $BC-D=0$ along the curve

$$Q_c^{(o)}(R,Pr,Pm)=\frac{1}{\pi^2}\left[\frac{Pm^2(1+Pr)}{Pr^2(1+Pm)}R - \left(\frac{(1+Pr)(Pr+Pm)}{Pr^2}\right)^{1/3}\left(\frac{Pm^2(1+Pr)}{Pr^2(1+Pm)}\right)^{2/3}R_c^{1/3}R^{2/3}\right] \qquad (2.15)$$

when $\{a,n\}=\{a_c^{(o)}(R,Pr,Pm),1\}$, where

$$a_c^{(o)}(R,Pr,Pm)=\left[\left(\frac{\pi^2Pm^2R}{2(1+Pm)(Pr+Pm)}\right)^{1/3}-\pi^2\right]^{1/2}. \qquad (2.16)$$

For all other $\{a,n\}$, $BC-D>0$ when $\{R,Q\}$ lies on (2.15).

When $\{R,Q\}$ is to the left of (2.15), $BC-D>0$ for all $\{a,n\}$, and when $\{R,Q\}$ is to the right of (2.15), $BC-D<0$ for some $\{a,n\}$. The curve $Q_c^{(o)}(R,Pr,Pm)$ starts on the R-axis at

$$R=R_s=\left(1+\frac{Pr+Pm+Pr\,Pm}{Pm^2}\right)R_c. \qquad (2.17)$$

This follows from setting $Q_c^{(o)}=0$ in (2.15). Clearly $R_s>R_c$ for all $Pr, Pm>0$. Thus, for all finite, non-zero Pr and Pm the curve $Q_c^{(o)}(R,Pr,Pm)$ starts on the R-axis to the right of the convection curve (2.11) since that one starts at $R=R_c$. Figure 2 shows this for two cases that are representative for all combinations of $Pr>0$ and $Pm>0$. The superscript (o) in (2.15) and (2.16) indicates that these are critical numbers for which there are overstable modes, as will be seen shortly.

Consider the factor

$$\delta=\frac{Pm^2(1+Pr)}{Pr^2(1+Pm)} \qquad (2.18)$$

which multiplies R in (2.15), and

$$\gamma=\left(\frac{(1+Pr)(Pr+Pm)}{Pr^2}\right)^{1/3}\left(\frac{Pm^2(1+Pr)}{Pr^2(1+Pm)}\right)^{2/3}, \qquad (2.19)$$

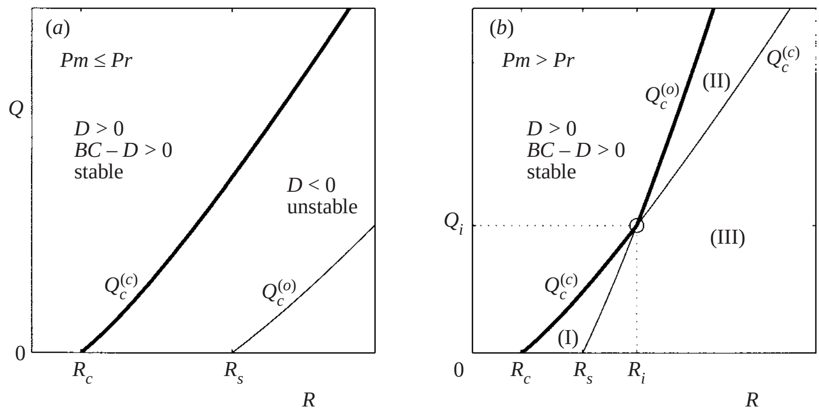

FIGURE 2. (*a*) Graph showing that when $Pm \leqslant Pr$ the overstability curve $Q_c^{(o)}(R, Pr, Pm)$ (2.15) does not cut the convection curve $Q_c^{(c)}(R)$ (2.11). The overstability curve starts at $R = R_s$ given by (2.17), the convection curve at $R = R_c$. To the left of $Q_c^{(c)}(R)$ (thick line) for all $\{a, n\}$ both $D > 0$ and $BC - D > 0$, implying stability. To the right $D < 0$ for some $\{a, n\}$ which implies instability (see table 1). This example was created with $Pm = 0.98$ and $Pr = 1$. (*b*) For $Pm > Pr$ the curves cut at $\{R, Q\} = \{R_i, Q_i\}$ given by (2.20). R_i and Q_i decrease with increasing Pm with Pr held fixed (see figure 3). In region (I) for some $\{a, n\}$ $D < 0$, $BC - D > 0$, in (II) $D > 0$, $BC - D < 0$ and in (III) $D < 0$, $BC - D < 0$, each combination implying instability (see table 1). This example was created using $Pm = 2$ and $Pr = 1$.

which multiplies $R_c^{1/3} R^{2/3}$ in (2.15). With a little algebra it follows that $\gamma > \delta$ for all finite $Pr, Pm > 0$. Further it is seen that $\delta > 1$ when $Pm > Pr$. Since for large positive R the convection curve is $Q_c^{(c)} \approx R/\pi^2$ while the overstability curve $Q_c^{(o)} \approx \delta R/\pi^2$, it follows that when $\delta > 1$ they must intersect at some point in the (R, Q)-plane. Equating (2.11) to (2.15), we find that the curves intersect at $\{R, Q\} = \{R_i, Q_i\}$ with

$$R_i(Pr, Pm) = \left(\frac{\gamma - 1}{\delta - 1}\right)^3 R_c, \quad Q_i(Pr, Pm) = \frac{1}{\pi^2}\left(\frac{\gamma - 1}{\delta - 1}\right)^2 \left[\left(\frac{\gamma - 1}{\delta - 1}\right) - 1\right] R_c. \tag{2.20}$$

When $Pm > Pr$, the terms multiplying R_c in (2.20) are positive because then $\gamma > \delta > 1$. Thus, the intersection occurs at $R_i > R_c$ and $Q_i > 0$. An example is shown in figure 2(*b*). For $Pr = Pm$ ($\delta = 1$), the intersection point $\{R_i, Q_i\}$ is formally at infinity. When $Pm < Pr$ we have $\delta < 1$ and the curves never intersect at any point in the positive quadrant of the (R, Q)-plane. A representative example is shown in figure 2(*a*).

The expressions (2.18) and (2.19) for δ and γ show that when we take the limit $Pm \to \infty$ while keeping Pr fixed, the term $(\gamma - 1)/(\delta - 1) \to 1$. Thus, with (2.20) it is seen that in this limit, $R_i \to R_c$ and $Q_i \to 0$. This also follows with (2.17), i.e. in the limit $Pm \to \infty$, $R_s \to R_c$. This limit is further discussed below in § 2.5. Figure 3 shows a representative example of how R_i and Q_i vary with increasing Pm while keeping Pr fixed.

2.4. *The marginal stability boundary*

When $Pm \leqslant Pr$ consider the 'boundary' consisting of the convection curve (2.11) drawn as a thick line in figure 2(*a*). For all points $\{R, Q\}$ to the left of this boundary $D > 0$ and $BC - D > 0$ for perturbations with any $\{a, n\}$. According to table 1 the system is stable for such R and Q values, i.e. the three roots of the cubic have $\mathrm{Re}\,\tilde{p} < 0$ for all $\{a, n\}$. From the discussion in § 2.2 it should be clear that for all $\{R, Q\}$ to

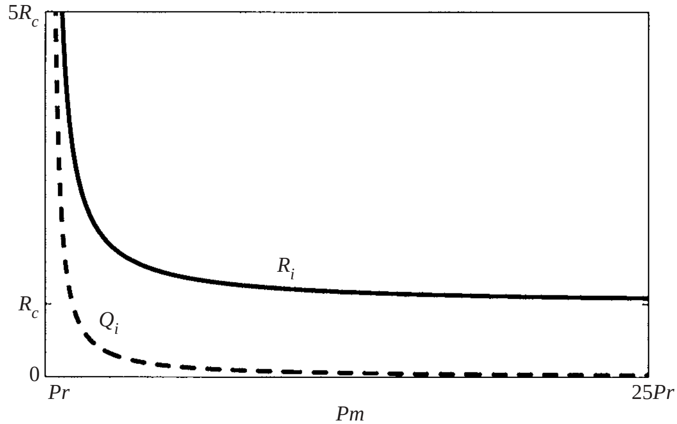

FIGURE 3. Example of $R_i(Pr, Pm)$ and $Q_i(Pr, Pm)$ given by (2.20) for fixed Pr and variable $Pm \geqslant Pr$. This graph was created using $Pr = 1$.

the right of the boundary there are perturbations with wavenumbers $\{a, n\}$ for which $D < 0$, which implies instability.

The boundary composed of the convection curve (2.11) for $R_c \leqslant R \leqslant R_i$ and the overstability curve (2.15) for $R > R_i$ separates the stable region from the unstable region in the (R, Q)-plane when $Pm > Pr$. This boundary is drawn as a thick line in figure 2(*b*). Again for all points $\{R, Q\}$ to the left of this boundary $D > 0$ and $BC - D > 0$ for perturbations with any $\{a, n\}$ and for such R and Q the system is stable. From the discussion in § 2.2 and § 2.3 it follows that in each of the regions marked as (I), (II) or (III) in figure 2(*b*) either $D < 0$ and/or $BC - D < 0$ for certain $\{a, n\}$. According to table 1 therefore for $\{R, Q\}$ to the right of the boundary the system is unstable.

When $\{R, Q\}$ lies on either boundary all modes are damped, except when $\{R, Q\}$ is on the convection curve in figure 2(*a*) (for $Pm < Pr$) or that section of the boundary in figure 2(*b*) (for $Pm > Pr$). In both cases a convective mode ($\tilde{p} = 0$) can be excited with $\{a, n\} = \{a_c^{(c)}(R), 1\}$. When $\{R, Q\}$ is on the overstability curve section in figure 2(*b*), two overstable modes can be excited with $\{a, n\} = \{a_c^{(o)}(R, Pr, Pm), 1\}$. At the intersection point $\{R_i, Q_i\}$ all modes are damped too, except when $\{a, n\} = \{a_c^{(c)}(R), 1\}$ or $\{a, n\} = \{a_c^{(o)}(R, Pr, Pm), 1\}$. There instability can set in either through stationary convection or through overstability, even simultaneously but with different wavelengths since $a_c^{(o)}(R, Pr, Pm) \neq a_c^{(c)}(R)$ for all finite non-zero Pr and Pm. This proves that for $Pm \leqslant Pr$ the marginal stability boundary is simply the convection curve (2.11) (figure 2*a*), while for $Pm > Pr$ it is the convection curve up to the intersection point $\{R_i, Q_i\}$ continued by the overstability curve (2.15) (figure 2*b*).

Finally, the non-dimensional oscillation frequency ω of the overstable modes along the overstability curve section of the marginal stability boundary is obtained by using the relations $\omega^2 = C$ and $\omega^2 = D/B$ (see Kloosterziel & Carnevale 2003). The common element R that appears in each of these expressions can be eliminated between them and the result is

$$\omega^2 = \frac{Pm - Pr}{Pm^2(1 + Pr)}\pi^2 Q - \frac{(\pi^2 + a^2)^2}{Pm^2}. \tag{2.21}$$

This equation was derived by Chandrasekhar in a somewhat complicated fashion. He used it to infer that for overstability it is necessary that

$$\pi^2 Q > \frac{1+Pr}{Pm-Pr}(\pi^2+a^2)^2$$

(equation (235) in Chandrasekhar 1961, §46) since this implies that ω is real, so that the roots $\tilde{p}=\pm i\omega$ are purely imaginary. He did not derive a formula for the frequency of the overstable modes. It is obtained by substituting $Q=Q_c^{(o)}(R, Pr, Pm)$ and $a=a_c^{(o)}(R, Pr, Pm)$ in (2.21). This yields

$$\omega^2(R, Pr, Pm) = \frac{Pm-Pr}{Pr^2(1+Pm)}R - \left(\frac{Pm^2-\frac{2}{3}Pr^2}{Pr^2Pm^2}\right)\left(\frac{Pm^2}{(1+Pm)(Pr+Pm)}\right)^{2/3} R_c^{1/3}R^{2/3}. \quad (2.22)$$

2.5. *The limit of zero magnetic diffusivity or infinite electrical conductivity*

According to (2.17), when $Pm \gg Pr$ and $Pm \gg 1$ the intersection point $\{R_i, Q_i\} \approx \{R_c, 0\}$. In other words, when Pm satisfies these two conditions, the marginal stability boundary is essentially everywhere the overstability curve so that instability can be expected to set in as oscillatory convection, even for very small Q and R slightly exceeding R_c. This limit $Pm \to \infty$ can be considered the limit $\eta \to 0$ or zero magnetic diffusivity. Chandrasekhar (1961) called this the limit of zero 'resistivity'. The stability properties of the system with $\eta=0$ can be determined by letting $Pm \to \infty$ in the coefficients B, C, D (2.2) of the cubic (2.1). But, some care must be exercised since $\sigma=(\mu\eta)^{-1}$, so that the electrical conductivity $\sigma \to \infty$ as $\eta \to 0$. Therefore with everything else held fixed, $Q \to \infty$ in C and D. But the terms proportional to Q/Pm that occur in C and D remain finite. The coefficients become

$$\left.\begin{aligned} B &= \frac{1+Pr}{Pr}(a^2+n^2\pi^2), \\ C &= \frac{1}{Pr}\left[(a^2+n^2\pi^2)^2 + n^2\pi^2 Pr\tilde{Q} - \frac{a^2R}{a^2+n^2\pi^2}\right], \\ D &= \frac{1}{Pr}(a^2+n^2\pi^2)n^2\pi^2\tilde{Q}, \end{aligned}\right\} \quad (2.23)$$

where

$$\tilde{Q} = \frac{\mu H^2 d^2}{\rho_0 \nu^2} = \frac{Q}{Pm}. \quad (2.24)$$

Alternatively, one can set $\eta=0$ in equation (102) in §42 of Chandrasekhar (1961), and by elimination between (101), (102) and (105), again derive a third-order differential equation in time for w. After substitution of the normal-modes ansatz for w as in §2 and subsequent non-dimensionalization of p with the viscous time scale d^2/ν, one also finds the coefficients (2.23).

First note that, unlike when Pm is finite ($\eta \neq 0$), $D>0$ everywhere when $H \neq 0$ so that for all Pr, R and $\tilde{Q}>0$ ($H \neq 0$) there is always one damped mode. $BC-D=0$ when

$$(x+n^2\pi^2)^3 + \frac{(x+n^2\pi^2)n^2\pi^2 Pr^2\tilde{Q}}{(1+Pr)} - xR = 0, \quad (2.25)$$

where $x=a^2$ as before. Equation (2.25) follows from (2.4) by replacing Q with $Pr^2\tilde{Q}/(1+Pr)$ in (2.4). Thus, in the $(R, \tilde{Q})$-plane the marginal stability boundary is the curve

$$\tilde{Q}_c^{(o)}(R, Pr) = \frac{1+Pr}{\pi^2 Pr^2}\left[R - R_c^{1/3} R^{2/3}\right]. \tag{2.26}$$

To the left of it $BC - D > 0$ and $D > 0$ for all $\{a, n\}$ (stability), whereas to the right of it $BC - D < 0$ for some $\{a, n\}$ while $D > 0$ (instability). For $\{R, \tilde{Q}\}$ on the curve $BC - D = 0$ when $\{a, n\} = \{a_c^{(o)}, 1\}$ with

$$a_c^{(o)}(R) = \left[(\pi^2 R/2)^{1/3} - \pi^2\right]^{1/2} \tag{2.27}$$

there will be two overstable modes since $D > 0$ for all $\{a, n\}$. Thus, for any $\tilde{Q} > 0$ there will be instability when R is large enough and the onset of instability is always through overstability. Remarkably, (2.27) is the same as the critical wavenumber for the onset of stationary convection (2.12) for given R and finite Pm whereas here it is the critical wavenumber for the onset of oscillatory convection, and unlike in the case of finite Pm (see (2.16)), it does not vary with the value of Pr. The frequency of the overstable modes is

$$\omega^2 = \frac{D}{B} = \frac{n^2 \pi^2 \tilde{Q}}{1+Pr}, \tag{2.28}$$

in which we have to put $n = 1$ and $\tilde{Q} = \tilde{Q}_c^{(o)}(R, Pr)$. This yields

$$\omega^2(R, Pr) = \frac{1}{Pr^2}\left[R - R_c^{1/3} R^{2/3}\right], \tag{2.29}$$

which, unlike the critical wavenumber (2.27), does depend on the Prandtl number.

The attentive reader may have noticed that (2.26) follows from (2.15) by dividing both sides in (2.15) by Pm and then taking the limit $Pm \to \infty$, while the same limit applied to (2.16) and (2.22) yields (2.27) and (2.29), respectively. But, if one had proceeded in this fashion one could have been led to believe that the convection curve (2.11) separating regions with $D > 0$ from $D < 0$ is still there, but it is not. As we showed, $D > 0$ everywhere when $\tilde{Q} > 0$.

It may seem strange that $D = 0$ for all $\{a, n\}$ when $\tilde{Q} = 0$ or $H = 0$, i.e. no magnetic field. This would appear to imply that for all R and $\{a, n\}$ there is convection, i.e. $\tilde{p} = 0$ is always a root. When $H = 0$ the problem should reduce to that of Rayleigh (1916) who showed that all modes are damped for all $\{a, n\}$ when $R < R_c$. The analysis here is based on the third-order differential equation with respect to time t for the vertical velocity w, derived by elimination between equations (101), (102) and (105) in chapter IV, § 42 of Chandrasekhar (1961). When $\eta = 0$ and $H = 0$ in that set of equations, one equation (equation (102) for the vertical component of the magnetic field perturbation) is decoupled from the other two (for w and the temperature perturbation) and is not needed. Thus, when $\eta = 0$ and $\tilde{Q} = 0$ the analysis should be based on an equation for w that is second-order in time. The root $\tilde{p} = 0$ for all R when $\tilde{Q} = 0$ is spurious and due to over-differentiation with respect to time.

3. Summary and discussion

Chandrasekhar (1961) sought to describe the convection curve in the (R, Q)-plane as the curve $R_c^{(c)}(Q)$ and the critical wavenumber for convection as $a_c^{(c)}(Q)$. For this he numerically solved the cubic in $x = a^2$ (2.6) as a function of Q and substituted the solution $a = a_c^{(c)}(Q)$ in (2.5) to obtain the critical Rayleigh number $R_c^{(c)}(Q)$. For a large range of Q-values the results were tabulated in table XIV in chapter IV. No closed-form formulae were noted by him for these relations that have Q as the

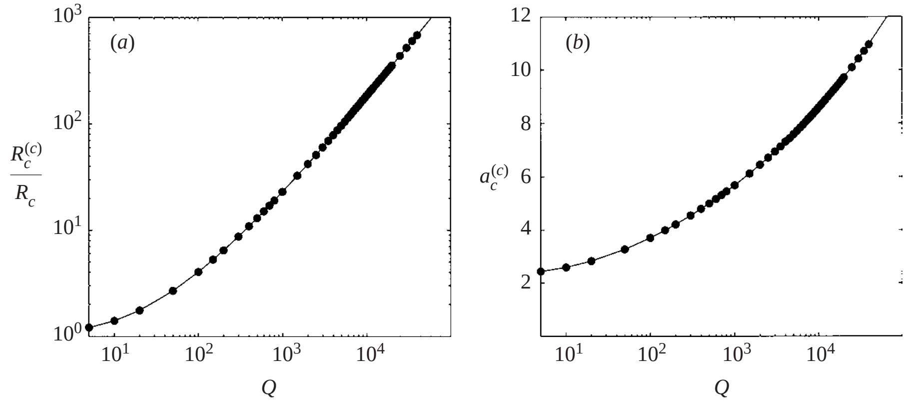

FIGURE 4. Graphs showing Chandrasekhar's data from table XIV, Chapter 4 (marked by •) with (a) $R_c^{(c)}(Q)$ normalized with R_c and (b) $a_c^{(c)}(Q)$. The curve drawn through the data points in (a) is the exact solution (3.2) and in (b) the solution (3.3). In (a) the same curve would be found when the inverse of (3.2) is plotted, i.e. $Q_c^{(c)}(R)$ given by (2.11).

independent variable. We found that the problem is equivalent to finding the critical Chandrasekhar number for which for given R the function $y = f(z)$ in (2.7) is tangent to the horizontal line $y = -n^2\pi^2 R$. This led to the discovery of simple expressions for the convection curve $Q_c^{(c)}(R)$ (2.11), the overstability curve $Q_c^{(o)}(R, Pr, Pm)$ (2.15), the critical horizontal wavenumber for the onset of stationary convection $a_c^{(c)}(R)$ (2.12) and oscillatory convection $a_c^{(o)}(R, Pr, Pm)$ (2.16) as well as the oscillation frequency $\omega(R, Pr, Pm)$ (2.22) of the overstable modes and the intersection point $\{R_i, Q_i\}$ (2.20) beyond which the overstability curve determines the stability boundary. None of these expressions appear to have been noted before.

Also, when Q is used as the independent variable, everything can be expressed with closed-form formulae. For example, a cubic equation in R describing the convection curve follows with (2.11):

$$R^3 - (3\pi^2 Q + R_c)R^2 + 3(\pi^2 Q)^2 R - (\pi^2 Q)^3 = 0. \tag{3.1}$$

Using the identity $4\cosh^3(z) - 3\cosh(z) = \cosh(3z)$, after a substitution which deletes the term proportional to R^2, the convection curve $R_c^{(c)}(Q)$ is found to be described by

$$R_c^{(c)}(Q) = R_c \left\{ \tfrac{1}{3} + Q' + \tfrac{2}{3}\left(1 + 6Q'\right)^{1/2} \cosh\left[\tfrac{1}{3}\operatorname{arcosh}\left(\frac{2 + 18Q' + 27Q'^2}{2(1 + 6Q')^{3/2}}\right)\right]\right\}, \tag{3.2}$$

where for convenience we have put $Q' = \pi^2 Q/R_c$. This is the inverse of the far simpler expression $Q_c^{(c)}(R)$ given by (2.11). An expression for $R_c^{(o)}(Q, Pr, Pm)$ (the overstability curve) follows from (3.2) by replacing R and Q with the substitutions given in (2.14). The cubic (2.6) can also be solved exactly to obtain the critical wavenumber for stationary convection as a function of the Chandrasekhar number. For $n = 1$ the solution is

$$a_c^{(c)}(Q) = \pi \left\{ \cosh\left[\tfrac{1}{3}\operatorname{arcosh}\left(1 + \frac{2Q}{\pi^2}\right)\right] - \tfrac{1}{2}\right\}^{1/2}. \tag{3.3}$$

This may be compared to the inverse relation $a_c^{(c)}(R)$ given by (2.12). In figure 4 we show the exact solutions $R_c^{(c)}(Q)$ and $a_c^{(c)}(Q)$ together with the 49 data points from Chandrasekhar's table XIV. We found that the relative errors in the data are extremely small, on the order of 1/100 of one percent. But, there are a number of $a_c^{(c)}$-values calculated by Chandrasekhar for large Q where the errors are almost 0.5%. The critical wavenumber $a_c^{(o)}(Q, Pr, Pm)$ for oscillatory convection follows by replacing Q in (3.3) by the transformed Q given in (2.14). This yields

$$a_c^{(o)}(Q, Pr, Pm) = \pi \left\{ \cosh\left[\tfrac{1}{3} \operatorname{arcosh}\left(1 + \frac{2Pr^2 Q}{\pi^2(1+Pr)(Pr+Pm)} \right) \right] - \tfrac{1}{2} \right\}^{1/2}. \quad (3.4)$$

If this is substituted in (2.21), i.e. setting $a = a_c^{(o)}(Q, Pr, Pm)$ there, one obtains a closed-form formula for the frequency $\omega(Q, Pr, Pm)$ of the overstable modes:

$$\omega^2(Q, Pr, Pm) = \frac{Pm - Pr}{Pm^2(1+Pr)} \pi^2 Q - \frac{\pi^4}{Pm^2} \left(\tfrac{1}{2} + \cosh\left[\tfrac{1}{3} \operatorname{arcosh}\left(1 + \frac{2Pr^2 Q}{\pi^2(1+Pr)(Pr+Pm)} \right) \right] \right)^2, \quad (3.5)$$

which is the inverse of the formula for $\omega^2(R, Pr, Pm)$ given by (2.22). The various asymptotic expressions established by Chandrasekhar for large Q follow quickly from the expressions just given and mentioned, using that for large argument z, $\operatorname{arcosh}(z) \approx \ln 2z$ and $\cosh(z) \approx \frac{1}{2} e^z$. They can also rapidly be determined using the exact expressions that have R as the independent variable, i.e. by assuming that R is large in (2.11), (2.12), (2.15), (2.16) and (2.22).

This work has been supported by Office of Naval Research grants N00014-97-1-0095 and N00014-96-1-0762 and National Science Foundation grants OCE 97-30843, OCE 01-28991 and OCE 01-29301.

REFERENCES

Chandrasekhar, S. 1952 On the inhibition of convection by a magnetic field. *Phil. Mag.* **43** (7), 501–532.

Chandrasekhar, S. 1961 *Hydrodynamic and Hydromagnetic Stability*. Oxford University Press.

Kloosterziel, R. C. & Carnevale, G. F. 2003 Closed-form linear stability conditions for rotating Rayleigh-Bénard convection with rigid, stress-free upper and lower boundaries. *J. Fluid Mech.* **480**, 25–42.

Thompson, W. B. 1951 Thermal convection in a magnetic field. *Phil. Mag.* **42** (7), 1417–1432.

Rayleigh, Lord 1916 On convection currents in a horizontal layer of fluid when the higher temperature is on the under side. *Phil. Mag.* **32** (6), 529–546.